THE LOGICAL WORK OF LEIBNIZ.

WHEN Leibniz's work is studied as a whole, as some of his remarks clearly show us that it ought to be studied,[1] we can see that his philosophy and his mathematics were founded in his logic. Although many have noticed the close connection of Leibniz's notions of his infinitesimal calculus and his monads, for example,[2] it was reserved for modern investigation to trace the complete story, both by reconstruction of Leibniz's thought and by taking into account hitherto unpublished documents written by Leibniz himself. This being the case, it is difficult not to complain of the way in which Leibniz's works have been published. Thus, Gerhardt, the editor of the most modern and complete collection of Leibniz's works, separated these works into "philosophical" and "mathematical." And yet Leibniz himself, in a letter to de l'Hôpital of December 27, 1694, had said: "My metaphysics is wholly mathematical";[3] and to Malebranche in March, 1699, he had said that "mathematicians need to be philosophers just as much as philosophers need to be mathematicians."[4]

In this article an attempt will be made to give an idea of Leibniz's logical work and plans for logical work. Great use has been made of Couturat's splendid book of 1901 mentioned in the Bibliography given above, but on some important points Couturat's account is supplemented. For example, this is so in the account (§ II) of the early ap-

[1] Cf. Couturat, 1901, pp. vii-ix.
[2] Thus cf. Latta, pp. 74-86.
[3] G. *math.*, Vol. II, p. 258.
[4] G., Vol. I, p. 356.

pearance of Leibniz's doctrine that the principle of identity held a very fundamental place in logic; in the sections (§§ IV, V) on the influence which guided Leibniz to a study of mathematics and on his mathematical work down to about the end of 1676; in the account (§ X) of the principle of continuity and its later developments; and in numerous footnotes throughout the paper. It cannot be too strongly emphasized that only these supplements are here treated at length, and that a knowledge of Couturat's book is assumed,—merely a tolerably full account of its contents has been given in the sections devoted to it.

I.

In a philosophical essay which Leibniz wrote in later life, under the name of "Gulielmus Pacidius," he said that when in his tenth year the library of his father, who was then dead, was thrown open to him, he seemed to be guided by the "*Tolle, lege*" of a higher voice, so that his natural thirst for knowledge led him to study the ancients and imbibe their spirit. "I burned," said he, "to get sight of the ancients, most of them known to me only by name, Cicero, Seneca, Pliny, Herodotus, Xenophon, Plato, and the historical writers, and many church fathers, Latin and Greek"; and soon it was with him as with "men walking in the sun, whose faces are browned without their knowing it."

It was characteristic of him to find some good in all he read.[5] "Like Socrates," he said, "I am always ready to learn";[6] and it was the study and spirit just mentioned, says Merz,[7] that led him to aim at two things which seemed to him to be foreign to the writers of the day: in words to attain clearness, and in matter usefulness.[8] The first aim led him to the study of logic, and, before he reached the age of twelve[9] he plunged with delight into the study

[5] G., Vol. VII, p. 526; Latta, pp. 1-2; Russell, pp. 5-7.

[6] Latta, p. 17.

[7] Merz, p. 13.

[8] *In verbis claritas, in rebus usus.*

[9] Couturat, 1901, pp. 33-34.

of scholastic logic.[10] He wrote out criticisms and plans for reform, and confessed that in later life he found great pleasure in re-reading his rough drafts written at the age of fourteen. At this age the idea occurred to him that just as the "predicaments" or categories of Aristotle serve to classify simple terms (concepts) in the order in which they furnish the matter of propositions, complex terms (propositions) might be classified in the order in which they furnish the matter of syllogisms, or of deduction generally. Neither he nor—probably—his teachers knew that that is exactly what geometricians do when they arrange their theorems in the order in which they are deduced from one another. Thus it was the mathematical method which was Leibniz's logical ideal even before he knew it, and it is not surprising that later on he took it as model and guide and grew to regard logic as a "universal mathematics."

Leibniz continued[11] to meditate on his idea of a classification of judgments about which his teachers had given him no information that was to the point, and it seems to have been in his eighteenth year that he arrived at thinking that all truths can be deduced from a small number of simple truths by analysis of the notions which are contained in them, and that all ideas can be reduced by decomposition to a small number of primitive and indefinable ideas. Thus we would only have to enumerate completely these simple ideas and thus form an "alphabet of human thoughts," and then combine them together, to obtain successively all complex ideas by an infallible process. This idea was a great joy to Leibniz, and while as a student of law at Leipsic University he was writing a dissertation on the necessity of introducing philosophical principles and reasoning into matters of law, and maintaining that the ancient jurists had brought so much thought and knowledge to bear upon their

[10] A short and good summary of the classical or syllogistic logic is given at *ibid.*, pp. 443-456.

[11] *Ibid.*, pp. 34-35.

subject that the principal task which they left to their successors was the systematic arrangement of the matter which they had collected, he was composing his treatise *De arte combinatoria.*[12] In it he showed that one of the principal applications of the art of combinations is logic, and more particularly the logic of discovery as opposed to demonstrative or syllogistic logic. The fundamental problem of the logic of discovery is, Given a concept as subject or predicate, to find all the proportions in which it occurs. Now, a proposition is a combination of two terms, a subject and a predicate. Thus the problem reduces to the problem of combinations.

In the latter part of this dissertation, Leibniz used and criticized the ideas of his predecessors, Raymond Lulle and others;[13] and one of the first applications given of the art of combinations was to the determination of the number of moods of the categorical syllogism.[14] Here I will draw attention to a relevant extract from a rather important manuscript of Leibniz. It was not referred to by Couturat, but is translated as the second of Leibniz's manuscripts on the infinitesimal calculus given in another article in this number of *The Monist.*

II.

When speaking of his early logical studies, Leibniz said in his *Historia et Origo*:[15] "When still a boy, when studying logic, he perceived that the ultimate analysis of truths that depend on reason reduces to these two things: definitions and identical truths; and that they alone of essentials are primitive and indemonstrable. And when it was objected to him that identical truths are useless and nugatory,

[12] Couturat, 1901, pp. 35-36; cf. also Merz, pp. 17-18, 106-114; Cantor, Vol. III, pp. 41-45.

[13] Couturat, 1901, pp. 36-39.

[14] On this and on Leibniz's later work on the syllogism, see *ibid.*, pp. 2-32.

[15] Cf. G., 1846, p. 4.

he showed the contrary by illustrations. Among other illustrations he showed that the great axiom that the whole is greater than the part could be demonstrated by a syllogism whose major premise was a definition and whose minor premise was an identical proposition. For, if of two things one is equal to a part of the other, the former is called the *less* and the latter the *greater*; let this be taken as the definition. Now, if to this definition we add the identical and undemonstrable axiom that everything possessed of magnitude is equal to itself, or A = A, then we have the syllogism:

"Whatever is equal to a part of another is less than that other (by definition);

"But the part is equal to a part of the whole (namely to itself, by identity);

"Therefore the part is less than the whole, Q. E. D."[16]

In another draft of the *Historia et Origo,* Leibniz speaks more at length about these early logical studies:[17] "Hitherto, while still a student, he was striving to bring logic itself to a certitude equal to that of arithmetic. He had observed from the first figure it was possible that the second and third might be deduced, not by employing conversion (which indeed itself seemed to him to need proof) but by employing solely the principle of contradiction; moreover, that conversions themselves could be demonstrated by the aid of the second and third figures by employing identical propositions; and, lastly, conversion being now demonstrated, by its aid the fourth figure could also be demonstrated; and thus that it was more indirect than the former (figures). Also he wondered very much at the force of these *identical truths,* for they were commonly considered to be nugatory and useless.[18] But later he perceived that the whole of arithmetic and geometry arose from identical

[16] Cf. Couturat, 1901, pp. 204-205. See also below, p. 590.

[17] G., 1846, p. 26, note 17.

[18] Cf. Couturat, 1901, pp. 8-12.

truths; and that, in general, all truths that were indemonstrable, if depending on pure reasoning, were identical; and that these combined with definitions produce identical truths. He gave an elegant example of this analysis in a demonstration of the theorem that the whole is greater than its part."

To Couturat's words[19] that Leibniz was concerned with showing the utility of identical propositions in reasoning and with defending them against the reproaches of insignificance and sterility urged against them by the empirical logicians, we may add two things: First, Leibniz seems to have held from early days the opinion that the foundations of logic are definitions and identical axioms;[20] secondly, in the *Historia* just mentioned, he traces to an identity his earliest mathematical discoveries in the summaton of series.

III.

We will now return to Leibniz's application of the art of combinations to the logic of discovery.[21] On analyzing all concepts by defining them—that is to say, by reducing them to combinations of simpler concepts— we arrive at a certain number of absolutely simple and indefinable concepts, and these "terms of the first order" are denoted by some such signs as numerals. "Terms of the second order" are obtained by combining in pairs those of the first order; and so on for terms of higher orders. Leibniz represented a compound term by the (symbolic) product of the numbers corresponding to the simple terms.

Leibniz was at that time still a novice in mathematics, and that explains many of the imperfections of the dissertation on the art of combinations; but still this early work

[19] *Ibid.*, p. 12.

[20] G., Vol. V, p. 92; Russell, pp. 17-19, 169; Couturat, 1901, p. 203. Cf. also the analogous example quoted from Leibniz and criticized by Frege, *Die Grundlagen der Arithmetik*, Breslau, 1884, pp. 7-8; Couturat, 1901, pp. 203, 205-207.

[21] Couturat, 1901, pp. 39-50.

contains the germ of his whole logic, which was with him a life-long study. That Leibniz was then a novice in mathematics comes out in the fact that he did not at first imagine his logic as a sort of algebra, but, since he was probably influenced by contemporary schemes, as a universal language or script.[22] This he had mentioned in his dissertation of 1666, and he developed it in the following years, especially from 1671 onward.[23] His "rational script" was, he says, a most powerful instrument of reason, and that it would promote commerce between nations should be esteemed the least of its uses. The notations or "characters" of a "real characteristic" represents ideas immediately and not words for them; thus, Egyptian and Chinese hieroglyphics and the symbols used by the alchemists for denoting substances are "real characters," and so they can be read off in various tongues; and, further, the "rational language" is formed on philosophical principles and is a help to *reasoning*.

IV.

According to Gerhardt[24] and Couturat,[25] Leibniz was led by logical investigations to the study of mathematics. About[26] the middle of the seventeenth century the study of mathematics in the universities of Germany was in a very bad state; and it is possibly enough to mention that his teachers were Johann Kühn and Erhard Weigel at the universities of Leipsic and Jena respectively. Still, Weigel seems to have gained a certain respect from Leibniz, and to have influenced him.[27] However, the facts that Leibniz had entered into correspondence with such men as Otto von Guericke and the learned Jesuit Honoratus Fabri of

[22] *Ibid.*, pp. 51-80. [23] *Ibid.*, pp. 59-61. [24] G., 1848, p. 7; G., 1855, p. 53.
[25] *Op. cit.*, p. 279. This is of course based on Leibniz's own statements.
[26] For the rest of this section, cf. G., 1848, pp. 7-9; G., 1855, pp. 53-54.
[27] Cf. Latta, p. 3.

Rome, and had sent the two parts of his *Hypothesis physica nova* to the lately founded learned Societies at London and Paris, show that Leibniz's active spirit was by no means satisfied with the knowledge he obtained in his university career. Before 1671 he had to depend almost entirely on books which came by chance into his hands, and thus it was that he was only acquainted with the beginnings of mathematical science and was for the most part ignorant of the progress made by the French, British and Italians during the seventeenth century. Also we must remember that he then considered law and history as his life-studies and thus only studied mathematics rather by the way and without any special industry. However these studies were very important for Leibniz, for he always kept in view their connection with logical researches and thus obtained exercise in expressing concepts by general signs. His first mathematical and philosophical writing of 1666 bears this character, and Leibniz himself repeatedly referred to it in the controversy about the discovery of the calculus.

In a letter[28] written from Mainz in the autumn of 1671 to the Duke of Brunswick-Lüneburg Leibniz announced a list of discoveries and plans for discoveries, arrived at by means of this new logical art, in natural science, mathematics, mechanics, optics, hydrostatics, pneumatics, and nautical science, not to speak of new ideas in law, theology and politics. Among these discoveries was that of a machine for performing more complicated operations than that of Pascal—multiplying, dividing, and extracting roots, as well as adding and subtracting.[29]

For Leibniz's mathematical education his stay in Paris, where he went in March of 1672 on a political mission, is

[28] Klopp, Vol. III, pp. 253ff.

[29] Sorley, p. 419. In G., 1848, p. 17; Latta, p. 6; and Merz, p. 53, it is implied that this machine was invented at Paris. This was also implied by Leibniz himself in § I of the article below on Leibniz's manuscripts relating to the infinitesimal calculus; but see Couturat, 1901, pp. 295-296. On the machine, see Cantor, Vol. III, p. 37.

of the greatest importance. Here for the first time he came into contact with the most eminent men of science of the time, and especially with Huygens who had presided over the French Academy since the year 1666. When Huygens published his celebrated *Horologium oscillatorium,* he sent a copy to Leibniz as a present. Leibniz saw from this work how very ignorant he was of mathematics, and his ambition to excel in this science flared up. In scientific conversations with Huygens the properties of numbers came into discussion, and Huygens, perhaps to test the talent of his new pupil, proposed to him the problem of finding the sum of a decreasing series of fractions whose numerators are unity and whose denominators are the triangular numbers. Leibniz found the correct result.[30]

Leibniz's intercourse with Huygens was interrupted by a journey to London in January of 1673.[31] In London, just as in Paris, he sought out the acquaintanceship of the celebrated men of England who lived in the capital. He had been in correspondence since 1670 with Henry Oldenburg, the secretary of the Royal Society, and met the mathematician Pell at the house of the chemist Robert Boyle. The conversation turned on the properties of numbers and Leibniz mentioned that he possessed a method of summing series of numbers by the help of their differences. When he explained himself more fully about this, Pell remarked that the method was contained in a book of Mouton called *De diametris apparentibus Solis et Lunae.* Leibniz had hitherto not known of this work; he borrowed it at once from Oldenburg, turned over its pages, and found that Mouton had obtained the same result in another way, and that his own method was more general.[32] By Pell Leibniz's attention was drawn to Mercator's *Logarithmotechnia,*

[30] G., 1848, pp. 17-19; G., 1855, p. 54.

[31] Cf. Cantor, Vol. III, p. 30.

[32] See the letter of Leibniz of February 3, 1673, to Oldenburg (G. *math.,* Vol. I, pp. 24ff).

especially because of the quadrature of the equilateral hyperbola contained in it, and Leibniz took this work with him to Paris. After his return to Paris he began, under Huygens's guidance, the study of the whole of higher mathematics. The *Géométrie* of Descartes, which hitherto he had known only superficially, the *Synopsis geometrica* of Honoratus Fabri, the writings of Gregory St. Vincent, and the letters of Pascal on the cycloid, were his guides.[33]

V.

We have another and rather different version of the way in which Leibniz was led to the study of mathematics. It was when he began to study at Leipsic University, which he entered in 1661—his fifteenth year—that he first became acquainted with the modern thinkers who had revolutionized science and philosophy.[34] "I remember," said Leibniz, "walking alone, at the age of fifteen, in a wood near Leipsic called the Rosenthal, to deliberate whether I should retain the doctrine of substantial forms. At last mechanism triumphed and induced me to apply myself to mathematics."[35]

In a letter of 1669 to Jacob Thomasius, one of his former teachers of philosophy at the University of Leipsic, Leibniz contended that the mechanical explanation of nature by magnitude, figure and motion alone is not inconsistent with the doctrines of Aristotle's Physics, in which he found more truth than in the Meditations of Descartes. Yet these qualities of bodies, he argued in 1668, require an incorporeal principle for their ultimate explanation. In 1671 he issued a *Hypothesis physica nova,* in which,

[33] G., 1848, pp. 19-20; G., 1855, pp. 54-55. On Leibniz's mathematical work of about this time, see Cantor, Vol. III, pp. 76-84, 115-118, 161-168, 179-184, 187-189, 191-216, 320-321; G., 1848, p. 15; G., 1855, pp. 33, 37-38, 48; G. 1846, p. xii, and the manuscripts on the calculus translated below; and Merz, pp. 50, 54-62. On the subsequent controversies to which this work gave rise, see Merz, pp. 84-96, 94-99, and Vol. III of Cantor.

[34] Latta, pp. 2-3.

[35] Cf. Merz, pp. 14-15; Latta, p. 3.

agreeing with Descartes that corporeal phenomena should be explained from motion, he contended that the original of this motion is a fine ether which constitutes light and, by penetrating all bodies in the direction of the earth's axis, produces the phenomena of gravity, elasticity and so on. The first part of the essay on concrete motion was dedicated to the Royal Society of London; the second part, on abstract motion, to the French Academy.[36]

VI.

It was in 1676 that Leibniz[37] seems first to have dreamed of a language which should at the same time be a calculus or algebra of thought, and then he definitely borrowed from mathematics his logical ideal.

But he soon found[38] that the construction *a priori* of a "rational language" was not so simple as he had believed at first, and in 1678 set about a comparative study of living languages for the purpose of extracting and combining the simple ideas expressed in them and of founding a "rational grammar";[39] and this language was by no means to be a calculus.[40]

Leibniz's problems then were, first, to make an inventory of human knowledge in which all known truths were to be demonstrated by reducing them to simple and evident principles, and, secondly, to invent signs to express the primitive concepts and their combinations and relations.[41] The second part was called the problem of the "Universal Characteristic"[42]—the characters being both what he called "real" and useful for reasoning, like the signs of arithmetic and algebra,—and the first part that of the "demonstrative encyclopedia."[43]

[36] Sorley, p. 419. On Leibniz's view of nature as a mechanism and his philosophy, cf. also Merz, pp. 41-43, 67-68, 72-73, 137-190.

[37] Couturat, 1901, pp. 61-62.

[38] *Ibid.*, pp. 63-64.

[39] *Ibid.*, pp. 64-79.

[40] *Ibid.*, pp. 78-79.

[41] *Ibid.*, pp. 79-80.

[42] *Ibid.*, pp. 81-118.

[43] *Ibid.*, pp. 119-175.

It was Leibniz who seems to have been the first to point out explicitly that "a part of the secret of analysis consists in the characteristic, that is to say, in the art of making a good use of one's notations,"[44] and we know,[45] both from his great step in inventing a supremely good notation and calculus for differentials and integrals and from the way in which he spoke of it from the very first, that he had the philosopher's property of being conscious of the help given to analysis by the invention of a *calculus of mathematical operations*— not "quantities"—which was very analogous to the calculus of ordinary algebra. The accusations that Leibniz had stolen ideas for an infinitesimal method are not only mistaken but also irrelevant. Leibniz himself said, without much exaggeration, that all his mathematical discoveries arose merely from the fact that he succeeded in finding symbols which appropriately expressed quantities and their relations.[46] In this connection we may mention that from Leibniz's Characteristic proceeded, besides his infinitesimal calculus and his dyadic arithmetic,[47] the use of a certain numerical notation in algebra and especially in the solution of simultaneous algebraic equations, the analogy between the development of, say, a binomial expansion and the repeated differentiation of a product of two factors, so that integration may be regarded as the operation of differentiation with a negative exponent, and so on.[48]

VII.

Leibniz formulated the conditions of a good Characteristic,[49] and clearly realized that it forms the basis for an

[44] Letter of 1693; Couturat, 1901, p. 83.
[45] Cf. *ibid.*, pp. 83-87.
[46] G. *math.*, Vol. VII, p. 17; Couturat, 1901, p. 84; Russell, p. 283.
[47] This is considered in another article in the present number.
[48] Couturat, 1901, pp. 473-500; Cantor, Vol. III, pp. 110-112, 230.
[49] Couturat, 1901, pp. 87-89.

algebra of logic, a *calculus ratiocinator* in which the rules of reasoning are translated by laws like those of algebra, and reasoning becomes a machinelike calculating process which frees the imagination where its action is not essential and thus increases the power of the mind.[50] With this tendency to economy of thought we may, it would seem, connect the opinion which Leibniz held on the value of the reduction of geometrical reasoning to analysis. "What is best and most convenient," said he,[51] "about my new (infinitesimal) calculus is that it offers truths by a kind of analysis and without any effort of imagination, which often only succeeds by chance, and that it gives us over Archimedes all the advantages which Vieta and Descartes had given us over Apollonius."

VIII.

The elaboration of the encyclopedia presupposed the knowledge of a universal method which should be applicable to all sciences, a "general science."[52] Little by little, the great plan for the encyclopedia, which occupied Leibniz at intervals from his twentieth year up to the time of his death, gave place gradually to the more restricted project of "beginnings of the general science," in which Leibniz would have exposed the principle of his method, that is to say his whole logic—which was an art of discovering as well as one of judging and demonstrating. All deduction, so Leibniz contended, rests on definitions, identical propositions; and so all truths can be demonstrated except identical and empirical propositions.[53] A definition is "nominal" when it indicates certain distinctive characters of the thing defined, so as to permit us to distinguish it from any other;

[50] *Ibid.*, pp. 96-103. Cf. on this point Jourdain, *Quart. Journ. Math.*, Vol. XLI, pp. 324-325, 329-332. Cf. also Russell, pp. 170, 206-208, 283-284.

[51] G. *math.*, Vol. II, p. 104; Russell, p. 283.

[52] Couturat, 1901, pp. 176-282. Cf. Latta, pp. 206-207.

[53] Couturat, 1901, pp. 184-188.

but a definition is only "real" when it shows the possibility or the existence of the thing. Indeed, the geometrical method requires that we demonstrate the possibility or ideal existence of every one of the figures defined either by indicating its construction or otherwise, so that every definition implies a theorem.[54]

Since the thorough analysis of truths and notions is the ideal of science, it is important to demonstrate the axioms, that is to say, to reduce them to definitions and identical propositions.[55] Indeed, every truth, whether necessary or contingent, is a relation of logical inclusion which can be discovered by simple analysis of the terms.[56]

Another part of Leibniz's logic is formed by questions arising out of the calculus of probabilities: the logic of probabilities is the science of temporal and contingent truths, and was, for Leibniz, a natural complement of the logic of certitude. And with this are connected considerations on the method of the natural sciences and the art of discovery.[57]

This art of discovery was regarded by Leibniz as his greatest discovery. He had cultivated it from his youth; it was to penetrate its secrets that he studied mathematics, because the sciences grouped together under that name were then the only ones in which this art was known and applied; and it was by trying to perfect it that he made all his mathematical discoveries. Thus we see why Leibniz's logic, mathematics, and philosophy were so closely connected, and also why Leibniz tried to give to philosophy a mathematical form.[58] But to extend the mathematical method to all sciences, the very idea of mathematics must be generalized, and this generalization resulted in the "Uni-

[54] *Ibid.*, pp. 188-195. On this theory of definitions and Leibniz's classification of ideas, see *ibid.*, pp. 195-200.

[55] *Ibid.*, pp. 200-207.

[56] *Ibid.*, pp. 208-213. On other principles (sufficient reason, and so on), see *ibid.*, pp. 213-239.

[57] *Ibid.*, pp. 239-278.

[58] *Ibid.*, pp. 278-282.

versal Mathematics,"[59] whence arose a general logic of relations.[60]. But the only algebra which Leibniz developed at all was what may be called attempts at a "logical calculus," dealing with the relations of identity and inclusion,[61] and the "geometrical calculus," dealing with the direct study of figures and spatial relations.[62] Both are particular applications of the Characteristic, and both are essays in Universal Mathematics.

We know now,[63] from Leibniz's manuscripts, that he possessed almost all the principles of the logic of Boole and Schröder, and on certain points he was further advanced than Boole. The chief reason why Boole succeeded where Leibniz failed is that Boole made the calculus of logic rest on the exclusive consideration of extension—and not intension—of concepts.

In criticism of the main points of Leibniz's logic Couturat[64] has advanced the following remarks. The postulates of Leibniz's logic are two in number: (1) All our ideas are compounded out of a small number of simple ideas; (2) Complex ideas proceed from these simple ideas by uniform and symbolical combination analogous to arithmetical multiplication. With regard to (1), the number of simple ideas is very much greater than Leibniz believed. With regard to (2), logical "multiplication" is not the only operation of which concepts are susceptible: we have to consider also logical "addition" and "negation." Leibniz, because he did not take account of negation, could not explain how simple ideas—which are all compatible with one another—can generate, by combination, mutually contradictory or exclusive complex ideas. Further, even if Leib-

[59] *Ibid.*, pp. 283-322. [60] *Ibid.*, pp. 300-318.

[61] *Ibid.*, pp. 323-387. These attempts began in 1679.

[62] *Ibid.*, pp. 388-430; Cantor, Vol. III, pp. 33-36; cf. also Couturat, 1901, pp. 529-538. A special article by Mr. A. E. Heath on the relation of Grassmann's ideas to Leibniz's will appear in the January issue.

[63] *Ibid.*, pp. 386-387. [64] *Ibid.*, pp. 431-441.

niz had succeeded in building up an algebra of classical logic, the logic of relations would still have remained outside. Leibniz was conscious of this and with him are to be found the first attempts at such a logic, but he did not go far, owing, it would seem, to an excessive respect for the authority of Aristotle.

We must always remember that, in his *Nouveaux essais,* Leibniz[65] laid stress on the importance of the invention of the form of syllogisms, and remarked that it is "a kind of universal mathematics whose importance is not sufficiently known"; and that he also remarked[66] that there are good asyllogistic conclusions, such as "Jesus Christ is God, therefore the mother of Jesus Christ is the mother of God," and "if David is the father of Solomon, without doubt Solomon is the son of David."

X.

We will now consider Leibniz's "law of continuity" and its later fortunes.

Leibniz, in the course of his letter of 1687 to Pierre Bayle on a general principle useful in the explanation of the laws of nature[67] says: "It [the principle] is absolutely necessary in geometry, but it succeeds also in physics, because the sovereign wisdom, which is the source of all things, acts as a perfect geometer, following a harmony to which nothing can be added.... It may be enunciated thus: 'When the difference of two cases can be diminished below every given magnitude in the data or in what is posited, it must also be possible to diminish it below every given magnitude in what is sought or in what results'; or to speak more familiarly: 'When the cases (or what is given) continually approach and are finally merged in each

[65] G., Vol. V, p. 460; Russell, p. 282; U., p. 266; Couturat, 1901, p. 1.

[66] G., Vol. V, p. 461; Russell, p. 283.

[67] G., Vol. III, pp. 51-55; Russell, pp. 64, 222. Cf. Cantor, Vol. III, pp. 277-278, 367; G., Vol. IV, p. 229; Couturat, 1901, pp. 233-237; Latta, pp. 37-39, 71, 83-84, 376-377.

other, the consequences or events (or what is sought) must do so too.' Which depends again on a still more general principle, namely: 'When the data form a series, so do the consequences (*datis ordinatis etiam quaesita sunt ordinata*).'"

Later on Leibniz also expressed his "law of continuity" by saying that "nature never makes leaps,"[68] and it would certainly appear that each of the above forms of the law implies the other. We first find an exact treatment of the question with Bolzano, and this will be mentioned presently.

Couturat[69] remarked on the first form that the enunciation was quite mathematical and that the principle was evidently suggested to Leibniz by his work on the infinitesimal calculus, "of which the first postulate is that we have to do with functions that are *continuous* and have derivatives." However this may be, it is a fact that the phrase "a function is subject to the law of continuity" used to mean throughout the eighteenth century and the first few years of the nineteenth, that the function in question was not one of those which Euler maintained could appear in the integrals of partial differential equations and which are expressed by differential equations in different intervals.[70]

For the moment I will distinguish with Arbogast between the "contiguity" and "continuity" of a function—the word "continuous" being used in the sense of Euler and the word "contiguous" in the sense in which we now, after Bolzano and Cauchy, use the word "continuous," and which seems to be the sense in which Leibniz used the phrase "varying according to the law of continuity." The

[68] G., Vol. V, p. 49. Cf. the passages quoted in Russell, pp. 222-223, and the first of the grounds against extended atoms mentioned on p. 234. Cf also *ibid.*, pp. 63-66.

[69] Couturat, 1901, p. 235 note.

[70] Cf., for example, Jourdain, in *Isis*, Vol. I, 1914, pp. 669-700.

fact then seems to be that Leibniz and his immediate successors thought that every function which could appear in analysis, geometry, or mathematical physics, was continuous and therefore contiguous; Euler made it probable that some important functions were not continuous and some of these were contiguous and some not. Fourier showed convincingly that those functions which seemed discontinuous to Euler were really continuous, since they could be represented by trigonometrical series, and thus that discontinuity was no mar to continuity. Finally, in 1814, Cauchy freed the language of analysis from the difficulty that one and the same function could be both discontinuous and continuous according to the way in which it was represented, by ignoring the notion of continuity and keeping only that of contiguity. Cauchy, in 1814, spoke of contiguity as "continuity,"[71] and this will seem to us confusing only if we do not reflect that the name "continuous" could be used by another conception as its original bearer was deceased.

It is, by the way, somewhat remarkable that Fourier should, in spite of this discovery, have clung to Euler's idea of "continuity" of a function and should have left to Cauchy the formulation of that useful property of certain functions which we still, like Cauchy, call "continuity"; but such is the fact. Especially at the beginning of his career, Cauchy was greatly influenced by the work of Fourier, and we may describe a great part of Cauchy's work by saying that it was the precise description and introduction into pure mathematics of many of the new ideas to which Fourier was led. Though we see the germs of a new conception of the "continuity" of a function in a paper by Cauchy of 1814, the conception was precisely defined by him only in 1821, and it is to Bernard Bolzano—who seems to have been uninfluenced by Fourier and very much in-

[71] *Ibid.*, pp. 688, 689, 690.

fluenced by Leibniz—that the priority of a precise formulation of the new conception of the "continuity" of a function must be attributed.

In a paper published in 1817,[72] Bolzano criticized the statement that, because a function "varies according to the law of continuity," it must pass through all intermediate values before it can attain to a higher one, on two grounds. In the first place, this is a provable theorem,—if, as he seems tacitly to imply, the following "correct" definition of "continuity" is used. In the second place, in the above statement "an incorrect conception of *continuity* is taken as basis. According to a *correct explanation* of the conception of continuity, we understand by the phrase: 'a function $f(x)$ varies according to the law of continuity for all values of x which lie inside or outside certain limits,' only that, if x is any such value, the difference $f(x - \omega) - f(x)$ can be made smaller than any given magnitude if ω may be taken as small as we wish."

XI.

In somewhat close connection with the work of Leibniz on mathematical logic stands the work of Johann Heinrich Lambert,[73] who sought—not very successfully—to develop the logic of relations. Toward the middle of the nineteenth century, George Boole[74] independently worked out and published his famous calculus of logic, which is almost exactly what Leibniz would have called a *calculus ratiocinator*. At the same time as Boole, and independently of him or of anybody else, Augustus De Morgan began to work out logic as a calculus, and later on, taking as his guide the maxim that logic should not consider merely certain kinds of deduction but deduction quite generally,

[72] See the further account and references, *ibid.*, pp. 695-697.

[73] See the historical parts of John Venn's *Symbolic Logic*, London, 1881; 2d ed., 1894, quoted by Jourdain, *Quart. Journ. of Math.*, Vol. XLI, p. 332.

[74] Cf. Jourdain, *loc. cit.*, pp. 332-352.

founded all the essential parts of the logic of relations. William Stanley Jevons[75] criticized and popularized Boole's work; and Charles S. Peirce, Richard Dedekind,[76] Ernst Schröder, Hermann and Robert Grassmann, Hugh MacColl,[77] John Venn, and many others, either developed the work of Boole and De Morgan or built up systems of calculative logic in modes which were largely independent of the work of others.

But it was in the work of Gottlob Frege, Guiseppe Peano, Bertrand Russell, and Alfred North Whitehead, that we find a closer approach to the *lingua characteristica* dreamed of by Leibniz. To this work other articles in this number will be devoted.

PHILIP E. B. JOURDAIN.

FLEET, HANTS, ENGLAND.

[75] Cf. Jourdain, *loc. cit.*, Vol. XLIV, pp. 113-128.
[76] Cf. *Monist* for July, 1916, pp. 415-427.
[77] Cf. Jourdain, *loc. cit.*, Vol. XLIII, pp. 219-236.